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Distributed Formation Control While Preserving Connectedness*

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Abstract— **This paper addresses the connectedness issue in multi-agent coordination, i.e. the problem of ensuring that the group stays connected while achieving some performance objective. In particular, we continue our previous work on rendezvous control and extend it to the formation control problem over dynamic interaction graphs. By adding appropriate weights to the edges in the graphs, we guarantee that the graphs stay connected, while achieving the desired formations.**

Index Terms— **Multi-Agent Coordination; Formation Control, Graph Laplacian, Connected Graphs.**

I. INTRODUCTION

As a fundamental problem in the distributed multi-agent coordination area, formation control has received considerable attentions. Leader based formation control has been studied for a long time, where either a real agent [2] or a virtual agent [3], [4], [15] is chosen as the leader. Recently, as graph theory has been successfully used for solving the rendezvous (or agreement) problem [18], [7], [9], [12], [16], [14], [1], a variety of graph-based formation control strategies have been proposed [5] and the references therein, most of which are leaderless.

A challenge faced in most multi-agent applications is that the agents are subjected to limited sensing and communication capabilities. Therefore, the underlying information network, consisting of sensing and communication links among the agents, might become disconnected, i.e. broken up into non-interacting sub-groups, if the control law does not take this limitation into account. For example, flocking under switching topologies was studied in [14], [19] and the references therein, where artificial potential functions or non-smooth Lyapunov functions were constructed over the graph structure. In [11], [12], state-dependent dynamic graphs were studied from a combinatoric point-of-view. A measure of local connectedness of a network was introduced in [17], depending entirely on the local interactions. In [20], connectivity constraints were related to individual agent's motion by the construction of a dynamically changing adjacency matrix.

In our previous work [8], the connectedness problem was investigated in the context of the rendezvous problem. In particular, connectedness was preserved by applying nonlinear weights on edges, which resulted in a weighted graph Laplacian. Based on a variation of these results, this paper extends the weighted graph Laplacian technique to the formation control problem.

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II. BACKGROUND

The problem that we are investigating in this paper is how to implement formation control in a distributive way while preserving connectedness with only limited information. Given N agents, whose positions $x_1, \ldots, x_N$ take on values in $\mathbb{R}^n$, we assume that the dynamics of each individual agent is given by a single integrator

$$\dot{x}_i = u_i,\ i = 1, \ldots, N. \tag{1}$$

We, moreover, associate an interaction graph with the available information flow in such a way that the nodes correspond to agents, and edges to available, inter-agent communication links. Such interaction graphs are thus reflective of the underlying network topology. It is worth of noticing that such graphs are of dynamic nature in that the existence of an edge between two nodes is affected by their relative positions.

By a Static Interaction Graph (SIG) $\mathcal{G} = (V, E)$, we understand the graph where the nodes $V = \{v_1, \ldots, v_N\}$ are associated to the different agents and the static edge set $E \subset V \times V$ is a set of unordered pairs of agents, with $(v_i, v_j) = (v_j, v_i) \in E$ if and only if a communication link exists between agents i and j. We will use the shorthand $V(\mathcal{G})$ and $E(\mathcal{G})$ to denote the edge and node sets associated with a graph $\mathcal{G}$.

Given an agent i, we will associate $\mathcal{N}_{\mathcal{G}}(i) = \{j \mid (v_i, v_j) \in E(\mathcal{G})\}$ with the neighborhood set to i, i.e. the set of agents adjacent to agent i. Using this terminology, what we understand by a limited information, time-invariant, decentralized control law is that $u_i = \sum_{j \in \mathcal{N}_\sigma(i)} f(x_i - x_j)$, where $\mathcal{N}_\sigma(i) \subseteq \mathcal{N}_{\mathcal{G}}(i)$. The symmetric indicator function $\sigma(i,j) = \sigma(j,i) \in \{0, 1\}$ determines whether or not the information available through edge (v_i, v_j) should be taken into account, with $j \in \mathcal{N}_\sigma(i) \ \Leftrightarrow\ (v_i, v_j) \in E(\mathcal{G}) \wedge \sigma(i,j) = 1$. (Using the terminology in [10], just because two nodes are "neighbors" it doesn't follow that they are "friends".) Along the same lines, the decentralized control law $f(x_i - x_j)$ is assumed to be anti-symmetric, i.e.

$$f(x_i - x_j) = -f(x_j - x_i),\ \forall\ (v_i, v_j) \in E(\mathcal{G}). \tag{2}$$

The type of control terms presented above have appeared repeatedly in the multi-agent coordination community, and an intuitive, linear control law for solving the rendezvous problem is given by

$$\begin{cases} \sigma(i,j) = 1 \\ f(x_i - x_j) = -(x_i - x_j) \end{cases} \quad \forall\ (v_i, v_j) \in E(\mathcal{G}),$$

which gives that

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_\mathcal{G}(i)} (x_i - x_j),\ i = 1,\ldots,N. \quad (3)$$

Under the dynamics in Equation (3), it has been shown that all agents approach the same point asymptotically, provided that the SIG is connected. And, even though this is a well-established result (see for example [9]), we will here outline a proof in order to establish some needed notation and tools.

Now, if the total number of edges is equal to M, and we associate an index with each edge such that $E(\mathcal{G}) = \{e_1,\ldots,e_M\}$, then the $N \times M$ incidence matrix of $\mathcal{G}^o$ is $\mathcal{I}(\mathcal{G}^o) = [\iota_{ij}]$, where

$$\iota_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the head of } e_j \\ -1 & \text{if } v_i \text{ is the tail of } e_j \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Through this incidence matrix, we can now define the *graph Laplacian* $\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{N\times N}$ as

$$\mathcal{L}(\mathcal{G}) = \mathcal{I}(\mathcal{G}^o)\mathcal{I}(\mathcal{G}^o)^T, \quad (5)$$

where we have removed the orientation dependence in the left hand side of Equation (5). The reason for this is that the Laplacian does not depend on the particular choice of orientation.

The graph Laplacian has a number of well-studied properties, found for example in [6], and we here list the properties of importance to the developments in this paper:

1) $\mathcal{I}(\mathcal{G}^o)\mathcal{I}(\mathcal{G}^o)^T = \mathcal{I}(\mathcal{G}^{o'})\mathcal{I}(\mathcal{G}^{o'})^T$ for all orientation o, o', i.e. the Laplacian is orientation-independent;
2) $\mathcal{L}(\mathcal{G})$ is symmetric and positive semidefinite;
3) The number of zero eigenvalues of $\mathcal{L}(\mathcal{G})$ equals to the number of connected components in $\mathcal{G}$;
4) If $\mathcal{G}$ is connected then null$(\mathcal{G})$ = span$\{\mathbf{1}\}$, where null$(\cdot)$ denotes the null-space.

If we now let the n-dimensional position of agent i be given by $x_i = (x_{i,1},\ldots,x_{i,n}),\ i = 1,\ldots,N$, and let $x = (x_1^T,\ldots,x_N^T)^T$, we can define the componentwise operator as $c(x,j) = (x_{1,j},\ldots,x_{N,j})^T \in \mathbb{R}^N,\ j = 1,\ldots,n$. Using this notation, together with the observation that Equation (3) can be decoupled along each dimension, we can in fact rewrite Equation (3) as

$$\frac{d}{dt}c(x,j) = -\mathcal{L}(\mathcal{G})c(x,j),\ j = 1,\ldots,n. \quad (6)$$

Now, as pointed out in [9] and [6], if $\mathcal{G}$ is connected then the eigenvector corresponding to the semi-simple eigenvalue 0 is $\mathbf{1}$. This, together with the non-negativity of $\mathcal{L}(\mathcal{G})$ and the fact that span$\{\mathbf{1}\}$ is $\mathcal{L}(\mathcal{G})$-invariant, is sufficient to show that $c(x,j)$ approaches span$\{\mathbf{1}\}$ asymptotically.

This result, elegant in its simplicity, can in fact be extended to dynamic graphs as well. In fact, since $c(x,j)^Tc(x,j)$ is a Lyapunov function to the system in (3), for any connected graph $\mathcal{G}$, the control law in Equation (6) drives the system to span$\{\mathbf{1}\}$ asymptotically *as long as* $\mathcal{G}(t)$ *is connected for all* $t \geq 0$.

This well-known result is very promising since dynamic network graphs are frequently occurring in that all real sensors and transmitters have finite range. This means that information exchange links may appear or be lost as the agents move around. In fact, if we focus our attention on Δ-disk proximity graphs, where edges are established between nodes v_i and v_j if and only if the agents are within distance Δ of each other, i.e. when $\|x_i - x_j\| \leq \Delta$, we get the Dynamic Interaction Graph (DIG) $\mathcal{G}(t) = (V, E(t))$, where $(v_i, v_j) = (v_j, v_i) \in E(t)$ if and only if $\|x_i(t) - x_j(t)\| \leq \Delta$.

The success of the control in Equation (3) hinges on an assumption that it shares with most graph-based results, e.g. [7], [19], namely on the connectedness assumption. Unfortunately, this property has to be assumed rather than proved.

What we will do for the remainder of this paper is to show how this assumption can be overcome by modifying the control law in Equation (3) in such a way that connectedness holds for all times, while ensuring that the control laws are still based solely on local information.

III. Connectedness Preserving Rendezvous

A. Static Graph

First, we will study the behavior of multi-agent system with a fixed network topology. In other words, the interaction graph will be of the SIG type, and we will show how the introduction of nonlinear edge-weights can be used to establish certain invariance properties.

To arrive at the desired invariance properties, we will first investigate decentralized control laws of the form

$$\begin{array}{l} \sigma(i,j) = 1 \\ f(x_i - x_j) = -w(x_i - x_j)(x_i - x_j) \end{array} \quad \forall (v_i, v_j) \in E(\mathcal{G}), \quad (7)$$

where $w : \mathbb{R}^n \to \mathbb{R}_+$ is a positive, symmetric weight function that associates a strictly positive and bounded weight to each edge in the SIG.

This choice of decentralized control law gives

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_\mathcal{G}(i)} w(x_i - x_j)(x_i - x_j), \quad (8)$$

which can be rewritten as

$$\frac{d}{dt}c(x,j) = -\mathcal{I}^o\mathcal{W}(x)\mathcal{I}^{oT}c(x,j),\ j = 1,\ldots,n, \quad (9)$$

where $\mathcal{W}(x)$ =diag$(w_1(x),\ldots,w_M(x)) \in \mathbb{R}^{M\times M}$, where, as before $M = |E(\mathcal{G})|$) is the total number of edges, and where we have associated an identity $(1,\ldots,M)$ to each of the edges.

We can then define the state-dependent, weighted graph Laplacian as

$$\mathcal{L}_\mathcal{W}(x) = \mathcal{I}^o\mathcal{W}(x)\mathcal{I}^{oT}, \quad (10)$$

where, as before, $\mathcal{W}(x) \in \mathbb{R}^{M\times M}$ is a diagonal matrix with each element corresponding to a strictly positive edge weight. It is moreover straightforward to establish that as long as the graph is connected, $\mathcal{L}_\mathcal{W}(x)$ is still positive semidefinite, with only one zero eigenvalue corresponding to the null-space span$\{\mathbf{1}\}$.

What we would like to show is that, given a critical distance δ together with the appropriate edge-weights, the edge-lengths never goes beyond δ if they start out being less than $\delta - \epsilon$, for some arbitrarily small $\epsilon \in (0, \delta)$. For this, we need to establish some additional notation, and, given an edge $(v_i, v_j) \in E(\mathcal{G})$, we let $\ell_{ij}(x)$ denote the edge vector between the agents i and j, i.e. $\ell_{ij}(x) = x_i - x_j$.

We moreover define the ϵ-interior of a δ-constrained realization of a SIG, $\mathcal{G}$, as

$$\mathcal{D}^{\epsilon}_{\mathcal{G},\delta} = \{x \in \mathbb{R}^{nN} \mid \|\ell_{ij}\| \leq \delta - \epsilon \ \forall (v_i, v_j) \in E(\mathcal{G})\}.$$

An edge-tension function $\mathcal{V}_{ij}$, can then be defined as

$$\mathcal{V}_{ij}(\delta, x) = \begin{cases} \frac{\|\ell_{ij}(x)\|^2}{\delta - \|\ell_{ij}(x)\|} & \text{if } (v_i, v_j) \in E(\mathcal{G}) \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Note that this edge-tension function (as well as its derivatives) is infinite when $\|\ell_{ij}(x)\| = \delta$ for some i, j, and, as such, it may seem like an odd choice. However, as we will see, we will actually be able to prevent the energy to reach infinity, and instead we will study its behavior on a compact set on which it is continuously differentiable.

The total tension energy of $\mathcal{G}$ can now be defined as

$$\mathcal{V}(\delta, x) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathcal{V}_{ij}(\delta, x). \tag{12}$$

Lemma 3.1: Given an initial position $x_0 \in \mathcal{D}^{\epsilon}_{\mathcal{G},\delta}$, for a given $\epsilon \in (0, \delta)$. If the SIG $\mathcal{G}$ is connected then the set $\Omega(\delta, x_0) := \{x \mid \mathcal{V}(\delta, x) \leq \mathcal{V}(\delta, x_0)\}$ is an invariant set to the system under the control law

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_{\mathcal{G}}(i)} \frac{2\delta - \|\ell_{ij}(x)\|}{(\delta - \|\ell_{ij}(x)\|)^2}(x_i - x_j). \tag{13}$$

Proof: We first note that the control law in Equation (13) can be rewritten as

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_{\mathcal{G}}(i)} \frac{\partial \mathcal{V}_{ij}(\delta, x)}{\partial x_i} = -\frac{\partial \mathcal{V}(\delta, x)}{\partial x_i} = -\nabla_{x_i} \mathcal{V}(\delta, x).$$

This expression may be ill-defined since it is conceivable that the edge-lengths approach δ and what will be shown is that this will not happen. In fact, assume that at time τ we have $x(\tau) \in \mathcal{D}^{\epsilon'}_{\mathcal{G},\delta}$ for some $\epsilon' > 0$. Then the time derivative of $\mathcal{V}(\delta, x(\tau))$ is

$$\begin{aligned} \dot{\mathcal{V}}(\delta, x(\tau)) &= \nabla_x \mathcal{V}(\delta, x(\tau))^T \dot{x}(\tau) = -\sum_{i=1}^{N} \dot{x}_i(\tau)^T \dot{x}_i(\tau) \\ &= -\sum_{j=1}^{n} c(x(\tau), j)^T \mathcal{L}_{\mathcal{W}}(\delta, x(\tau))^2 c(x(\tau), j), \end{aligned} \tag{14}$$

where $\mathcal{L}_{\mathcal{W}}(\delta, x)$ is given in Equation (10), with weight positive definite (on $\Omega(\delta, x_0)$) matrix $\mathcal{W}(\delta, x)$

$$\begin{aligned} \mathcal{W}(\delta, x) &= \mathrm{diag}(w_k(\delta, x)), \quad k = 1, 2, \ldots, M, \\ w_k(\delta, x) &= \frac{2\delta - \|\ell_k(x)\|}{(\delta - \|\ell_k(x)\|)^2}, \end{aligned} \tag{15}$$

where we have arranged the edges such that subscript k corresponds to edge k. We will use this notation interchangeably with w_{ij} and ℓ_{ij}, whenever it is clear from the context.

Note that for any ϵ' bounded away from 0 from below and δ from above, and for any $x \in \mathcal{D}^{\epsilon'}_{\mathcal{G},\delta}$, the time derivative of the total tension energy is well-defined. Moreover, for any such x, $\mathcal{V}(\delta, x)$ is non-negative and $\dot{\mathcal{V}}(\delta, x)$ is non-positive (since $\mathcal{L}_{\mathcal{W}}(\delta, x)$ is positive semidefinite for all $x \in \Omega(\delta, x_0)$. Hence, in order to establish the invariance of $\Omega(\delta, x_0)$, all that needs to be shown is that, as $\mathcal{V}$ decreases (or at lest does not increase), no edge-distances will tend to δ. In fact, since $\mathcal{D}^{\epsilon}_{\mathcal{G},\delta} \subset \mathcal{D}^{\epsilon'}_{\mathcal{G},\delta}$ if $\epsilon > \epsilon'$, we would have established the invariance of $\Omega(\delta, x_0)$ if we could find an $\epsilon' > 0$ such that, whenever the system starts from $x_0 \in \mathcal{D}^{\epsilon}_{\mathcal{G},\delta}$, we can ensure that it never leaves the superset $\mathcal{D}^{\epsilon'}_{\mathcal{G},\delta}$.

Let $\hat{\mathcal{V}}_\epsilon := \max_{x \in \mathcal{D}^{\epsilon}_{\mathcal{G},\delta}} \mathcal{V}(\delta, x)$. This maximum always exists and is obtained when all edges are at the maximal allowed distance $\delta - \epsilon$, i.e. $\hat{\mathcal{V}}_\epsilon = \frac{M(\delta - \epsilon)^2}{\epsilon}$, which is a monotonously decreasing function in ϵ over $(0, \delta)$.

What we will show next is that we can bound the maximal edge distance that can generate this total tension energy, and the maximal edge-length $\hat{\ell}_\epsilon \geq \delta - \epsilon$ is one where the entire total energy is contributed from that one single edge. In other words, all other edges have length 0, and the maximal edge length satisfies $\hat{\mathcal{V}}_\epsilon = \frac{\hat{\ell}^2_\epsilon}{\delta - \hat{\ell}_\epsilon}$, i.e. $\frac{M(\delta - \epsilon)^2}{\epsilon} = \frac{\ell^2_\epsilon}{\delta - \ell_\epsilon}$, which implies that $\hat{\ell}_\epsilon \leq \delta - \frac{\epsilon}{M} < \delta$. Hence ℓ_ϵ is bounded away from above from δ and it is moreover bounded from above by a strictly decreasing function in ϵ on $(0, \delta)$. Hence, as $\mathcal{V}$ decreases (or at least is non-increasing), no edge-distances will tend to δ, which completes the proof. ■

The invariance of $\Omega(\delta, x_0)$ now leads us to the main SIG theorem.

Theorem 3.2: Given a connected SIG $\mathcal{G}$ with initial condition $x_0 \in \mathcal{D}^{\epsilon}_{\mathcal{G},\delta}$, for a given $\epsilon > 0$. Then the multi-agent system under the control law in Equation (13) asymptotically converges to a common point.

Proof: The proof of convergence is based on LaSalle's invariance theorem. Let $\mathcal{D}^{\epsilon}_{\mathcal{G},\delta}$ and $\Omega(\delta, x_0)$ be defined as before. From Lemma 3.1, we know that $\Omega(\delta, x_0)$ is positive invariant with respect to the dynamic in Equation (13). We also note that span$\{\mathbf{1}\}$ is $\mathcal{L}_{\mathcal{W}}(\delta, x)$-invariant for all $x \in \Omega(\delta, x_0)$. Hence, due to the fact that $\dot{\mathcal{V}}(\delta, x) \leq 0$, with equality only when $c(x(t), j) \in \text{span}\{\mathbf{1}\}$, $\forall j \in \{1, \ldots, n\}$, convergence to span$\{\mathbf{1}\}$ follows. ■

B. Dynamic Graphs

As already pointed out, during a maneuver, the interaction graph $\mathcal{G}$ may change as the different agents move in and out of each others sensory ranges. What we focus on in this section is whether or not an argument, similar to the previous stability result, can be constructed for the case when $(v_i, v_j) \in E(\mathcal{G})$ if and only if $\|x_i - x_j\| \leq \Delta$.

In fact, we intend to reuse the tension energy from the previous section, with the particular choice of $\delta = \Delta$. However, since in Equation (15) $\lim_{\|\ell_k\| \uparrow \Delta} w_k(\Delta, \|\ell_k\|) = \infty$, we can not directly let the inter-agent tension energy affect the dynamics as soon as two agents form edges in between them, i.e. as they move within distance Δ of each other. The reason for this is that we can not allow infinite tension energies in the definition of the control laws. To overcome this problem, we chose to introduce a certain degree of hysteresis into the system, through the indicator function σ. In particular, let the total tension energy be affected by an edge (v_i, v_j) that was previously not contributing to the total energy, when $\|\ell_{ij}\| \leq (\Delta - \epsilon)$, where $\epsilon > 0$ is the predefined *switching threshold*. Once the edge is allowed to contribute to the total tension energy, it will keep doing so for all subsequent times. Note that the switching threshold can take on any arbitrary value in $(0, \Delta)$. The interpretation is simply that a smaller ϵ-value corresponds to a faster inclusion of the inter-robot information into the decentralized control law. In other words, what we propose for the Δ-disk proximity DIGs is thus to let

$$\begin{aligned} \sigma(i,j)[t^+] &= \begin{cases} 0 & \text{if } \sigma(i,j)[t^-] = 0 \ \wedge \ \|\ell_{ij}\| > \Delta - \epsilon \\ 1 & \text{otherwise} \end{cases} \\ f(x_i - x_j) &= \begin{cases} 0 & \text{if } \sigma(i,j) = 0 \\ -\frac{\partial \mathcal{V}_{ij}(\Delta, x)}{\partial x_i} & \text{otherwise,} \end{cases} \end{aligned} \tag{16}$$

where we have used the notation $\sigma(i,j)[t^+]$ and $\sigma(i,j)[t^-]$ o denote $\sigma(i,j)$'s value before and after the state transition.

Before we can state the rendezvous theorem for dynamic graphs, we also need o introduce the subgraph $\mathcal{G}_\sigma \subset \mathcal{G}$, induced by the indicator function σ: $\mathcal{G}_\sigma = (V(\mathcal{G}), E(\mathcal{G}_\sigma))$, where $E(\mathcal{G}_\sigma) = \{(v_i, v_j) \in E(\mathcal{G}) \mid \sigma(i,j) = 1\}$.

Theorem 3.3: Given an initial position $x_0 \in \mathcal{D}^{\epsilon}_{\mathcal{G}^0, \Delta}$, where $\epsilon > 0$ is the switching threshold in Equation (16), and where $\mathcal{G}^0$ is the initial Δ-disk DIG. Assume that the graph $\mathcal{G}^0_\sigma$ is connected, where $\mathcal{G}^0_\sigma$ is the graph induced by the initial indicator function value. Then, by using the control law

$$u_i = -\sum_{j \in \mathcal{N}_\sigma(i)} \frac{\partial \mathcal{V}_{ij}(\Delta, x)}{\partial x_i}, \tag{17}$$

where $\sigma(i,j)$ is given in Equation (16), the group of agents asymptotically converges to span$\{\mathbf{1}\}$.

Proof: Since, from Lemma 3.1, we know that no edges in $\mathcal{G}^0_\sigma$ ill be lost, only two possibilities remain, namely that no new edges will be added to the graph during the maneuver, or new edges will in fact be added. If no edges are added, then we know from Theorem 3.2 that the system will converge to span$\{\mathbf{1}\}$ asymptotically. However, the only graph consistent with $x \in$ span$\{\mathbf{1}\}$ is $\mathcal{G}^0_\sigma = K_N$ (the complete graph over N nodes), and hence no new edges will be added only if the initial, indicator induced graph is complete. If it is not complete, at least one new edge will be added. But, since $\mathcal{G}^0_\sigma$ is an arbitrary connected graph, and connectivity can never be lost by adding new edges, we get that new edges will be added until the indicator induced graph is complete, at which point the system converges asymptotically to span$\{\mathbf{1}\}$. ■

IV. Formation Control

In the previous section, the connectedness-preserving control method solves the rendezvous problem. In what follows, we will follow the same methodology to solve the distributed formation control problem.

A. Graph Formation

By formation control, we understand the problem of driving the collection of mobile agents to some translationally invariant target geometry, such that their relative positions satisfy some desired topological and physical constraints. These constraints can be described by a connected, edge-labelled graph $\mathcal{G}_d = (V, E_d, d)$, where the subscript d denotes "desired". Here, E_d encodes the desired robot interconnections, i.e. whether or not a desired inter-agent distance is specified between two agents or not, and the edge-labels $d : E_d \rightarrow \mathbb{R}^n$ defines the desired relative inter-agent displacements, with $\|d_{ij}\| < \Delta$ for all i, j such that $(v_i, v_j) \in E_d$. In other words, what we would like is that $x_i - x_j \rightarrow d_{ij}$ $\forall i, j$ such that $(v_i, v_j) \in E_d$.

One may notice that it is possible that the assignment of general edge-labels to a DIG may result in conflicting constraints. This is addressed in [13] as the realization problem of connectivity graphs. We will not discuss this problem here and simply assume that the constraints are compatible.

Given a desired formation, the goal of the distributed formation control is to find a feedback law such that:

F1) The dynamic interaction graph $\mathcal{G}(t)$ converges to a graph that i s a supergraph of the desired graph $\mathcal{G}_d$ (without labels) in finite time. In other words, what we want is that $E_d \subset E(t)$ for all $t \geq T$, for some finite $T \geq 0$;

F2) $\|\ell_{ij}(t)\| = \|x_i(t) - x_j(t)\|$ converges asymptotically to $\|d_{ij}\|$ for all i, j such that $(v_i, v_j) \in E_d$; and

F3) The feedback law utilizes only local information.

Here "F" stands for "formation" and what will be established is in fact that these properties hold for a particular choice of decentralized control law.

B. Graph-Based Formation Control

Analogous to the treatment of the rendezvous problem, we first propose a solution to the formation control problem, and then show that this solution does in fact preserve connectedness as well as guarantee convergence in the sense of F1 and F2 above. The solution will be based on a variation of the previously derived rendezvous controller. In fact, assume that we have established a set of arbitrary targets $\tau_i \in \mathbb{R}^n$ that are consistent with the desired inter-agent displacement, i.e.$d_{ij} = \tau_i - \tau_j$, $\forall\, i, j$ s.t. $(v_i, v_j) \in E_d$. We can then define the displacement from τ_i at time t as $y_i(t) = x_i(t) - \tau_i$. As before, we let $\ell_{ij}(t) = x_i(t) - x_j(t)$ and we moreover let $\lambda_{ij}(t) = y_i(t) - y_j(t)$, implying that $\lambda_{ij}(t) = \ell_{ij}(t) - d_{ij}$.

Now, under the assumption that $\mathcal{G}_d$ is a connected *spanning graph* of the initial interaction graph $\mathcal{G}$, i.e. $V(\mathcal{G}_d) = V(\mathcal{G})$ and $E_d \subseteq E(\mathcal{G})$, we propose the following control law:

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_{\mathcal{G}_d}(i)} \frac{2(\Delta-\|d_{ij}\|)-\|\ell_{ij}-d_{ij}\|}{(\Delta-\|d_{ij}\|-\|\ell_{ij}-d_{ij}\|)^2}(x_i-x_j-d_{ij}). \tag{18}$$

The reason why this seemingly odd choice makes sense is because we can again use the edge-tension function $\mathcal{V}$ to describe this control law. In particular, using the following parameters in the edge-tension function

$$\mathcal{V}_{ij}(\delta-\|d_{ij}\|,y) = \begin{cases} \frac{\|\lambda_{ij}\|^2}{\Delta-\|d_{ij}\|-\|\lambda_{ij}\|} & \text{if } (v_i,v_j)\in E_d \\ 0 & \text{otherwise,} \end{cases} \tag{19}$$

we obtain the decentralized control law

$$\begin{cases} \sigma(i,j)=1 \\ f(x_i-x_j) = -\frac{\partial\mathcal{V}_{ij}(\Delta-\|d_{ij}\|,y)}{\partial y_i} \end{cases} \quad \forall\ (v_i,v_j)\in E_d.$$

As such, along each individual dimension, the dynamics becomes $\frac{dc(x,j)}{dt} = \frac{dc(y,j)}{dt} = -\mathcal{L}_W(\Delta-\|d\|,y)c(y,j)$, where $j=1,2,\ldots n$ and $\mathcal{L}_W(\Delta-\|d\|,y)$ is the graph Laplacian associated with $\mathcal{G}_d$, weighted by $W(\Delta-\|d\|,y)$, and where we have used the convention that the term $\Delta-\|d\|$ should be interpreted in the following manner:

$$\begin{aligned} W(\Delta-\|d\|,y) &= \mathrm{diag}(w_k(\Delta-\|d_k\|,y)), \\ w_k(\Delta,-\|d_k\|,y) &= \frac{2(\Delta-\|d_k\|)-\|\lambda_k\|}{(\Delta-\|d_k\|-\|\lambda_k\|)^2}). \end{aligned} \tag{20}$$

where $k=1,2,\ldots|E_d|$. Here, again, the index k runs over the edge set E_d. Note that this construction allows us to study the evolution of y_i, rather than $x_i,\ i=1,\ldots,N$, and we formalize this in the following lemma for static interaction graphs:

Lemma 4.1: Let the total tension energy function be

$$\mathcal{V}(\Delta-\|d\|,y) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{V}_{ij}(\Delta-\|d_{ij}\|,y). \tag{21}$$

Given $y_0\in\mathcal{D}^{\epsilon}_{\mathcal{G}_d,\Delta-\|d\|}$, with $\mathcal{G}_d$ being a connected spanning graph, then the set $\Omega(\Delta-\|d\|,y_0) := \{y \mid \mathcal{V}(\Delta-\|d\|,y)\le\mathcal{V}_0\}$, where $\mathcal{V}_0$ denotes the initial value of the total tension energy function, is an invariant set under the control law in Equation (18), under the assumption that the interaction graph is static.

Proof: By the proposed control law in Equation (18),

$$\begin{aligned} \dot{y}_i &= -\sum_{j\in\mathcal{N}_{\mathcal{G}_d}(i)} \frac{\partial\mathcal{V}_{ij}(\Delta-\|d_{ij}\|,y)}{\partial y_i} \\ &= -\frac{\partial\mathcal{V}(\Delta-\|d\|,y)}{\partial y_i} = -\nabla_{y_i}\mathcal{V}(\Delta-\|d\|,y). \end{aligned} \tag{22}$$

The non-positivity of $\dot{\mathcal{V}}$ now follows the same argument as in Equation (14) in the proof of Lemma 3.1. Moreover, for each initial $y_0\in\mathcal{D}^{\epsilon}_{\mathcal{G}_d,\Delta-\|d\|}$, the corresponding maximal, total tension-energy induces a maximal possible edge length. Following the same line of reasoning as in the proof of Lemma 3.1, the invariance of $\Omega(\Delta-\|d\|,y_0)$ thus follows. ■

Note that Lemma 4.1 says that if we could use $\mathcal{G}_d$ as a SIG, $\Omega(\Delta-\|d\|,y_0)$ is an invariant set. In fact, it is straightforward to show that if $\mathcal{G}_d$ is a spanning graph to the initial proximity Δ-disk DIG, then it remains a spanning graph to $\mathcal{G}(x(t))\ \forall t\ge 0$.

Lemma 4.2: Given an initial condition x_0 such that $y_0=(x_0-\tau_0)\in\mathcal{D}^{\epsilon}_{\mathcal{G}_d,\Delta-\|d\|}$, with $\mathcal{G}_d$ being a connected spanning graph of $\mathcal{G}(x_0)$, the group of autonomous mobile agents adopting the decentralized control law in Equation (18) can guarantee that $\|x_i(t)-x_j(t)\| = \|l_{ij}(t)\| < \Delta,\ \forall t>0$ and $(v_i,v_j)\in E_d$.

Proof: Given two agents i,j that are adjacent in $\mathcal{G}_d$, and suppose that $\|\lambda_{ij}\|=\|y_i-y_j\|$ approaches $\Delta-\|d_{ij}\|$. Since $\mathcal{V}_{ij}\ge 0,\ \forall i,j$ and $t>0$, as well as $\lim_{\|\lambda_{ij}\|\uparrow(\Delta-\|d_{ij}\|)}\mathcal{V}_{ij}=\infty$, this would imply that $\mathcal{V}\to\infty$, which contradicts Lemma 4.1. As a consequence, $\|\lambda_{ij}\|$ is bounded away from $\Delta-\|d_{ij}\|$. This means that

$$\|\ell_{ij}\| = \|\lambda_{ij}+d_{ij}\| \le \|\lambda_{ij}\|+\|d_{ij}\| < \Delta-\|d_{ij}\|+\|d_{ij}\| = \Delta,$$

and hence edges in E_d are never lost under the control law in Equation (18). In other words, $\|l_{ij}(t)\|<\Delta,\ \forall t\ge 0$, which in turn implies that connectedness is preserved. ■

We have thus established that if $\mathcal{G}_d$ is a spanning graph of $\mathcal{G}(x_0)$ then it remains a spanning graph of $\mathcal{G}(x(t)),\ \forall t>0$ (under certain assumptions on x_0), even if $\mathcal{G}(x(t))$ is given by a Δ-disk DIG. And, since the control law in Equation (18) only takes pairwise interactions in E_d into account, we can view this dynamic situation as a static situation, with the SIG being given by $\mathcal{G}_d$. Now we need to verify the properties of F1, F2, and F3. That F3 (decentralized control) is satisfied follows trivially from the definition of the control law in Equation (18). Moreover, we have already established that F1 (finite time convergence to the appropriate graph) holds trivially as long as it holds initially, and what remains to be shown here is thus that we can drive the system in finite time to a configuration in which F1 holds, after which Lemma 4.2 applies. Moreover, we need to establish that the inter-robot displacements (defined for edges in E_d) converge asymptotically to the desired, relative displacements (F3), which is the topic of the next theorem.

Theorem 4.3: Under the same assumptions as in Lemma 4.2, $\|\ell_{ij}(t)\|=\|x_i(t)-x_j(t)\|$ converges asymptotically to $\|d_{ij}\|$ for all i,j such that $(v_i,v_j)\in E_d$.

Proof: Based on the observation that $\mathcal{G}_d$ remains a spanning graph to the DIG, together with the observation that

$$\frac{dc(y,j)}{dt} = -\mathcal{L}_W(\Delta-\|d\|,y)c(y,j), \quad j=1,2,\ldots n,$$

Theorem 3.2 ensures that $c(y,j)$ will converge to $\mathrm{span}\{\mathbf{1}\},\ \forall j\in\{1,\ldots,n\}$. What this implies is that all displacements must be the same, i.e. that $y_i=\zeta,\ \forall i\in\{1,\ldots,N\}$ for some constant $\zeta\in\mathbb{R}^n$. But, this simply means that the system converges asymptotically to a fixed translation away from the target points $\tau_i,\ i=1,\ldots,N$, i.e.

$$\lim_{t\to\infty} y_i(t) = \lim_{t\to\infty}\big(x_i(t)-\tau_i\big) = \zeta,\ i=1,\ldots,N,$$

which in turn implies that

$$\begin{aligned} \lim_{t\to\infty} \ell_{ij}(t) &= \lim_{t\to\infty} \big((x_i(t) - x_j(t)\big) \\ &= \lim_{t\to\infty} \big(y_i(t) + \tau_i - y_j(t) - \tau_j\big) \\ &= \zeta + \tau_i - \zeta - \tau_j = d_{ij}, \end{aligned} \tag{23}$$

$\forall i, j$ s.t. $(v_i, v_j) \in E_d$, which completes the proof. ■

C. Hybrid, Rendezvous-to-Formation Control Strategies

The last property that must be established is that it is possible to satisfy F1, i.e. that the initial Δ-disk proximity DIG does in fact converge to a graph that has $\mathcal{G}_d$ as a spanning graph in finite time. If this was achieved then Theorem 4.3 would be applicable and F2 (asymptotic convergence to the correct inter-agent displacements) would follow. To achieve this, we propose to use the rendezvous control law developed in the previous section for gathering all agents into a complete graph, of which trivially any desired graph is a subgraph. Moreover, we need to achieve this in such a manner that the assumptions in Theorem 4.3 are satisfied.

Let K_N denote the complete graph over N agents. Moreover, we will use K_N^ε to denote the DIG that is a complete graph in which no inter-agent distances are greater than ε. This notation is slightly incorrect in that graphs are inherently combinatorial objects, while inter-agent distances are geometric, and, to be more precise, we will use the notation $\mathcal{G} = K_N^\varepsilon$ to denote the fact that $\mathcal{G} = K_N$ and $\|\ell_{ij}\| \leq \varepsilon, \ \forall\ (i,j),\ i \neq j$. The reason for this construction is that, in order for Theorem 4.3 to be applicable, the initial condition has to satisfy $y_0 = (x_0 - \tau_0) \in \mathcal{D}^\epsilon_{\mathcal{G}_d, \Delta - \|d\|}$, which is ensured by making ε small enough. Moreover, since the rendezvous controller in Equation (17) asymptotically achieves rendezvous, it will consequently drive the system to K_N^ε in finite time, $\forall \varepsilon \in (0, \Delta)$.

After K_N^ε is achieved, the controller switches to the controller in Equation (18), as depicted in Figure 1. However, this hybrid control strategy is only viable if the condition that $\mathcal{G} = K_N^\varepsilon$ is locally verifiable in the sense that the agents can decide for themselves that a synchronous mode switch is triggered [10]. In fact, if an agent has $N - 1$ neighbors, i.e. degree $N - 1$, all of which are within a distance $\varepsilon/2$. Hence, when one agent detects this condition, it will trigger a switching signal, and the transition in Figure 1 occurs. Regardless of which, we know that this transition will in fact occur in finite time in such a way that the initial condition assumptions of Theorem 4.3 are satisfied.

V. Conclusion

In this paper, a graph-based nonlinear feedback control law is studied for distributed formation control. The nonlinear feedback law is based on weighted graph Laplacians and it is proved to be able to solve the formation control problem while preserving connectedness.

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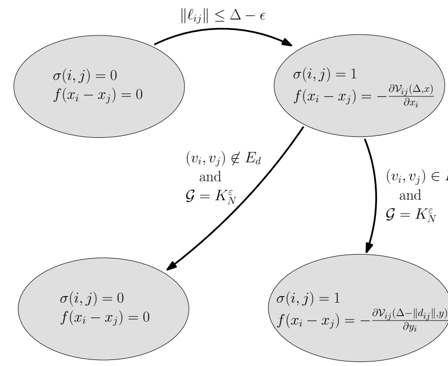

Fig. 1. The figure shows a state machine describing how the system undergoes transitions from rendezvous (collection of the agents to a tight, complete graph), to formation control.

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